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Nonlinear solution for an applied overpressure on a moving stream

L. W. SCHWARTZ

Department of Applied Mathematics, University of Adelaide, Adelaide, S.A. 5001, Australia.

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SUMMARY

A boundary-integral method is given for the numerical solution of the exact equations for steady two-dimensional potential flow past a fixed pressure distribution on the free surface of a fluid of infinite depth. The variation in wave-resistance coefficient with overpressure and Froude number is presented. A drag-free nonlinear profile is obtained.

1. Introduction

This paper presents a numerical procedure for the solution of steady two-dimensional potential flows past a fixed pressure distribution on the free surface of a fluid of infinite depth. The dynamic boundary condition on the free-surface is retained in its complete nonlinear form. Such a flow can be produced by blowing air onto the surface of water flowing in a channel with parallel sidewalls.

In general, a surface pressure distribution is a model for the hydrodynamic effects of an air-cushion vehicle or hovercraft. It may also be viewed as an inverse method of solution to the classical ship-wave problem. With a given distribution of overpressure, the surface will deform in some manner to be determined as part of the solution. The portion of the deformed surface upon which the pressure is applied can be viewed as a rigid obstacle that, when inserted in the water, will produce the calculated flow pattern. It is possible, at least conceptually, to solve for flow past different classes of hull forms by iterated solutions to the pressure-distribution problem.

The linearized version of the two-dimensional problem was solved long ago and is discussed in detail by Lamb [1]. A remarkable feature of the linearized solution is that certain pressure distributions will not produce any surface elevation at a distance; that is, their motion will not be accompanied by a train of waves. Thus, within the context of inviscid theory, the surface shapes induced by these pressure distributions will be drag-free hull forms.

More recently, Von Kerczek and Salvesen [2] have produced numerical solutions to the nonlinear problem in water of finite depth. Their method requires that a mesh of points be placed over the entire fluid region; thus they cannot treat the infinite depth case in principle. In practice their method is restricted to relatively modest values of the depth to pressure-distribution-length ratio.

In the present work, the problem is solving using a boundary integral technique based on Cauchy's integral formula. The vector of unknowns is used entirely to specify the shape of the free surface which consequently may be found to high accuracy. The method has been successfully used in a number of free-surface wave problems (e.g. [3], [4], [5]).

Basic theory is given in Section 2. A numerical algorithm involving a truncation of the doubly-infinite boundary integral and a low-order discretization of the governing equations is derived in Section 3. While higher-order schemes may be expected to improve the accuracy still further, it is a remarkable feature of the present method that simple numerical methods provide good results over most of the parameter range of interest. Results for various values of non-dimensional overpressure p_0 and pressure-distribution-length Froude number *Fr* are given in Section 4. The system of nonlinear algebraic equations derived in Section 3 is solved by a Newtonian iteration in each case. Errors resulting from truncation of the boundary integral are shown to be quite manageable. Wave resistance is determined as a function of *Fr* for two values of p_0 and compared with the predictions of linear theory. For a large value of overpressure, a nonlinear profile with a resistance coefficient of order 10^{-4} has been determined. Extension of the method to finite depth is discussed in the concluding section.

2. Theory

In laboratory coordinates, consider a uniform flow from left to right with speed U. The undisturbed fluid region occupies the lower half of the z-plane with the free surface corresponding to $y = Im(z) = 0$. Now subject the free surface to a pressure distribution P. Assuming P to vanish far upstream where the flow remains uniform, the dynamic free surface condition may be written

$$\frac{P}{\rho} + \tfrac{1}{2} q^2 + gy = \text{const} = \frac{U^2}{2}, \tag{1}$$

where q is the local fluid speed, ρ the constant fluid density and g the acceleration of gravity.

This steady flow must, in addition, satisfy the kinematic conditon that the free surface is a streamline.

Selecting U as reference speed and $U^2/2g$ as reference length, (1) becomes, in dimensionless variables,

$$p_0 + q^2 + y = 1 \tag{2}$$

where the pressure coefficient is defined as

$$p_0 = \frac{P}{\tfrac{1}{2}\rho U^2}.$$

The conditions of incompressibility and irrotationality of the fluid motion are satisfied by introducing the complex potential $f(z) = \phi(x,y) + i\psi(x,y)$. f is assumed to be analytic in the portion of the z-plane occupied by the fluid. Here ϕ is the velocity potential and ψ, the stream function.

The free surface is taken to be the streamline $\psi = 0$. A considerable simplification is achieved by solving the problem in the f- rather than the z-plane. The condition (2) is imposed on the known boundary $Im(f) = 0$ and the kinematic condition is satisfied identically. Applying Cauchy's integral formula for the analytic function $z'(f)$ on a contour enclosing the entire lower-half f-plane except for an infinitesimal semi-circle around the free-surface point $f = \phi$ readily yields

$$x'(\phi) = 1 - \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{y'(\varphi)\,\mathrm{d}\varphi}{\varphi - \phi}, \tag{3}$$

as a relation between the real and imaginary parts of z' on $\psi = 0$. The Cauchy principal value of the improper integral is to be taken.

The velocity term in (2) may be written

$$q^2 = \frac{1}{z'(f)\,\overline{z'(f)}} = \frac{1}{[x'(\phi)]^2 + [y'(\phi)]^2}, \tag{4}$$

where the bar signifies complex conjugation. For a given distribution of pressure $p_0(\phi)$, the system (2) – (4) determines a solution for the surface shape in parametric form $(x(\phi), y(\phi))$. The solution must also satisfy a radiation condition prohibiting upstream disturbances, i.e.

$$x \to \phi, \quad y \to 0 \quad \text{as} \quad \phi \to -\infty.$$

3. Numerical method

The system of equations (2) – (4) is to be satisfied at N equally spaced points on the surface. The integral in (3) is truncated both upstream and downstream subject to the requirement that the applied pressure distribution is sufficiently far removed from either end-point. The error produced by this truncation can be estimated by comparing solutions for different choices of endpoints. Let h be the point spacing in ϕ. Equation (3) is replaced by

$$x'_{j-\frac{1}{2}} = 1 - \frac{1}{\pi} \int_0^{Nh} \frac{y'(\varphi)\,\mathrm{d}\varphi}{\varphi - \phi_{j-\frac{1}{2}}}, \quad j = 1, \ldots, N. \tag{5}$$

The principal-value integral is treated simply by spacing points symmetrically with respect to the pole, viz. Monacella [6]. For simplicity, the numerical integration uses the trapezoidal rule; thus the right side of (5) becomes

$$1 - \frac{1}{\pi}\left[\frac{1}{2}\,\frac{y'_0}{0-(j-\frac{1}{2})} + \frac{y'_1}{1-(j-\frac{1}{2})} + \ldots + \frac{y'_{N-1}}{(N-1)-(j-\frac{1}{2})} + \frac{1}{2}\,\frac{y'_N}{N-(j-\frac{1}{2})}\right].$$

Values are specified at the zero-th mesh point as $y_0 = y'_0 = x_0 = 0$ and $x'_0 = 1$. The values at the Nth point, on the other hand, are to be determined as part of the solution. This asymmetry is the (numerical) radiation condition. The values of x' at the halfpoints are related to the whole-point values by linear interpolation,

$$x'_{j-\frac{1}{2}} = \frac{x'_{j-1} + x'_j}{2}, \quad j = 1, \ldots, N. \tag{6}$$

Equations (5) and (6) may be combined to yield

$$x'_1 = 1 - \frac{4}{\pi} \sum_{i=1}^{N-1} \frac{y'_i}{2(i-1)+1} - \frac{2}{\pi} \frac{y'_N}{2(N-1)+1}, \tag{7a}$$

$$x'_j = 2 - x'_{j-1} - \frac{4}{\pi} \sum_{i=1}^{N-1} \frac{y'_i}{2(i-j)+1} - \frac{2}{\pi} \frac{y'_N}{2(N-j)+1}, \quad j = 2, \ldots, N. \tag{7b}$$

Denote the local error in the surface condition by E_i; thus

$$E_i = p_{0i} + q_i^2 + y_i - 1 \tag{8}$$

where p_{0i} are the given overpressures.
Here y_i is found by trapezoidal integration of y' according to

$$y_i = \frac{h}{2} \sum_{j=1}^{i} (y'_{j-1} + y'_j). \tag{9}$$

Given an initial guess to the solution vector y'_i, an improved approximation $y'_i + \Delta y'_i$ is found by Newtonian iteration. That is, the corrections to y'_i are found as the solution to the system of linear equations

$$\sum_{j=1}^{N} \left(\frac{\partial E_i}{\partial y'_j}\right) \Delta y'_j = -E_i, \quad i = 1, \ldots, N. \tag{10}$$

The Jacobian may be computed in terms of local values by

$$\begin{aligned} \frac{\partial E_i}{\partial y'_j} &= \frac{\partial}{\partial y'_j}\left(\frac{1}{x_i'^2 + y_i'^2}\right) + \frac{\partial y_i}{\partial y'_j} \\ &= -2q_i^4 \left[x'_i \frac{\partial x'_i}{\partial y'_j} + y'_i \delta_{ij}\right] + \frac{\partial y_i}{\partial y'_j} \end{aligned} \tag{11}$$

where δ_{ij} is the Kronecker δ-function. The derivatives $\partial x'_i/\partial y'_j$ and $\partial y_i/\partial y'_j$ are found explicitly from (7) and (9).

The numerical procedure may be summarised as follows:
(i) for a given overpressure distribution, assume an initial set of values for y'_i, $i = 1, N$. In fact $y'_i = 0$ is usually adequate;
(ii) compute x'_i from (7);
(iii) compute q_i from (4) and y_i from (9);
(iv) find the error vector E_i defined in (8);

(v) compute the Jacobian in (11);
(vi) find the correction to the y_i' vector as the solution to the linear system in (10);
(vii) return to step (ii) and continue until the desired degree of convergence is achieved.

A FORTRAN program using this algorithm, and employing a packaged linear equation solver, requires only about 70 program steps. A single converged solution using 100 surface points typically requires five Newtonian iterations and 20 sec. of CPU time on a CYBER 173.

4. Discussion of results

The algorithm derived above has been used to find a number of nonlinear solutions corresponding to various values of overpressure p_0 and pressure distribution length L. Note that L is related to the length-based Froude number by

$$L = 2/(Fr)^2 .$$

The results to be presented are all for p_0 = constant over the length L. A number of other solutions corresponding to different distributions of pressure have been computed without difficulty; however constant pressure is considered to be the most important special case and has the incidental benefit that the linear solution, used for comparison, assumes a particularly simple form.

In Figure 1 we compare the computed nonlinear solution for $p_0 = .15$ for ϕ between 9.75 and 16.25. With $N = 100$ and $h = .5$, thirteen mesh points represent the loaded portion of the free surface. Four-point interpolation is used to find the x-values at the 'bow' x_b and 'stern' x_s

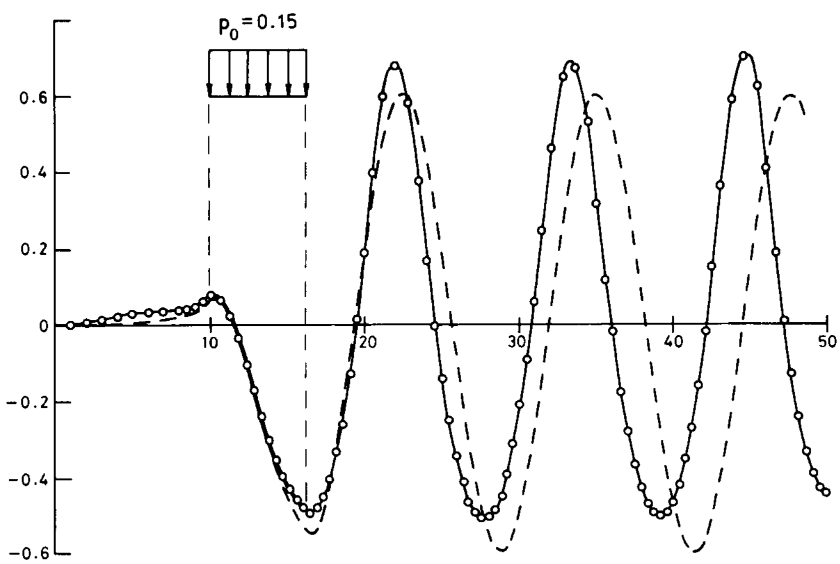

Figure 1. Comparison of linear and numerical solutions for $p_0 = .15$, $L = 6.35$, $Fr = .561$; o-o-o numerical, $h = .5$, $N = 100$, – – – – – linear.

to yield the values $L = x_s - x_b = 6.35$ and $Fr = .561$. Note that with the present normalization the length of a free linear wave is 4π.

For comparison, the profile computed by linear theory for these values of p_0 and Fr is also shown in the figure. Its equation is

$$y(x) = p_0 [Y(x; x_s) - Y(x; x_b)] \tag{12a}$$

where

$$Y(x_1, x_2) = \begin{cases} -2\cos\left(\frac{x_1 - x_2}{2}\right) + \frac{1}{\pi} f\left(\frac{x_1 - x_2}{2}\right); & x_1 > x_2 \\ -1 - \frac{1}{\pi} f\left(\frac{x_2 - x_1}{2}\right); & x_2 > x_1 . \end{cases} \tag{12b}$$

Here

$$f(x) = \int_0^\infty \frac{e^{-xt}\, dt}{t^2 + 1} \tag{12c}$$

is the auxiliary function to the sine and cosine integrals discussed in Abramowitz and Stegun [7]. For $0 < x < 1$, f may be found to high accuracy using a few terms in its convergent series expansion. For $x > 1$, Abramowitz and Stegun give an accurate approximation in the form of a rational fraction.

If R is the dimensional wave resistance (per unit width), a dimensionless resistance coefficient may be defined as

$$C = \frac{g\rho R}{2P^2} . \tag{13}$$

According to linear theory with constant overpressure P, this coefficient, C_ℓ say, is given simply as

$$C_\ell = 1 - \cos\frac{L}{2} = 1 - \cos\frac{1}{(Fr)^2} . \tag{14}$$

In the nonlinear case, no such simple formula is available but the coefficient may be computed from the converged solution as

$$C_n = \frac{y_b - y_s}{2p_0} \tag{15}$$

where y_b and y_s, the (dimensionless) elevations of the bow and stern, are also found by four-point interpolation.

The nonlinear solution in Figure 1 shows narrow crests and broad troughs for the downstream waves. At a distance of about one wavelength aft of the body, the wave train is essentially periodic and indistinguishable from the free wave in form. Note also that the mean water level has been displaced upward in the nonlinear wave train. Thus mass is convected by the waves, as predicted by nonlinear progressive wave theory, viz. Schwartz [8]. The downstream wave height/wavelength ratio is .106, about 75 per cent of the theoretical maximum in deep water. Slightly higher waves can be calculated for this value of Fr by increasing the overpressure slightly. Ultimately accuracy and convergence will be limited by the relatively small number of points per wave cycle and the tendency of the f-plane method to distribute points sparsely in regions of low speed.

The upstream 'infinity' condition has been imposed at $\phi = 0$, slightly less than one wavelength upstream of the body. Some inaccuracy can be expected near the ends of the integration interval. In Figure 1, the converged nonlinear solution incorrectly predicts a small elevation about one-half wavelength upstream of the bow. Also near the downstream truncation point, the nonlinear profile begins to deviate from the periodic wave train.

The linear solution overpredicts the draft or displacement under the pressure distribution by about 7 per cent. The linear wave resistance coefficient $C_\ell = 2.00$ is also high by this percentage compared with $C_n = 1.86$.

The question of up-and-downstream truncation error is addressed again in the results presented in Figure 2. Here, for $p_0 = .12$ and essentially the same Froude number as in the first example we compare two solutions with $N = 100$ and 160 respectively using the same point spacing $h = .5$. The longer run places 20 additional points upstream and 40 extra points downstream. The figure shows the entire region where the two runs have points in common. Significant differences can be seen between the two runs for about the first half wavelength upstream and the last whole wavelength at the downstream edge. To the extent that the longer run suffers analogous truncation error, the portion of it shown in the figure (as a solid line) may be considered exact. The calculated resistance coefficients for the two runs differ by less than 2 per cent. Here the wave-train steepness is .078 and the resistance coefficient $\simeq 1.90$.

The wave resistance is plotted versus overpressure length in Figure 3. The range of L is 2.6 to 27 with the corresponding length Froude number varying from .88 to .27. The linear theory

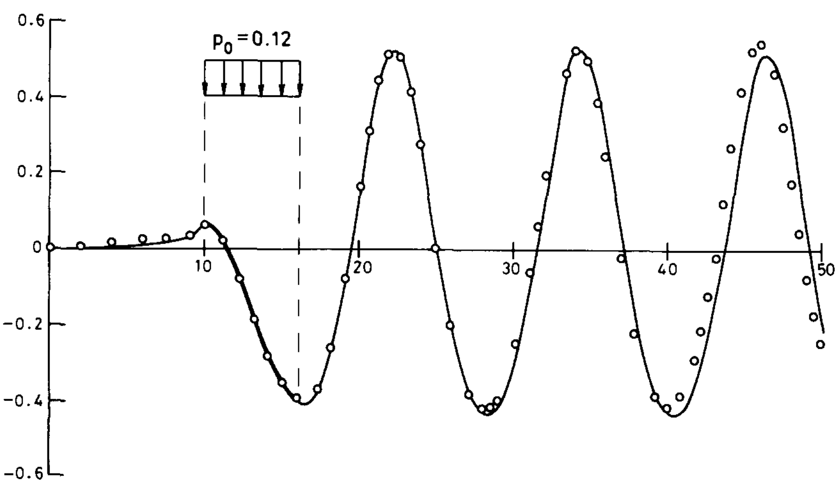

Figure 2. Comparison of numerical solutions showing effects of truncation, $p_0 = .12$, $L = 6.37$, $Fr = .560$; ——— $h = .5, N = 160$; ○ ○ $h = .5, N = 100$.

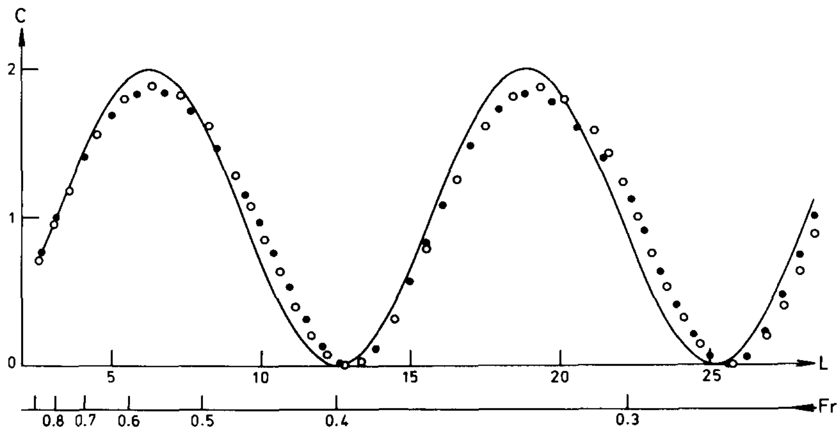

Figure 3. Wave-resistance coefficient versus pressure length and Froude number; —— linear theory; ○ ○ ○ nonlinear $p_0 = .1$, ● ● ● nonlinear $p_0 = .15$.

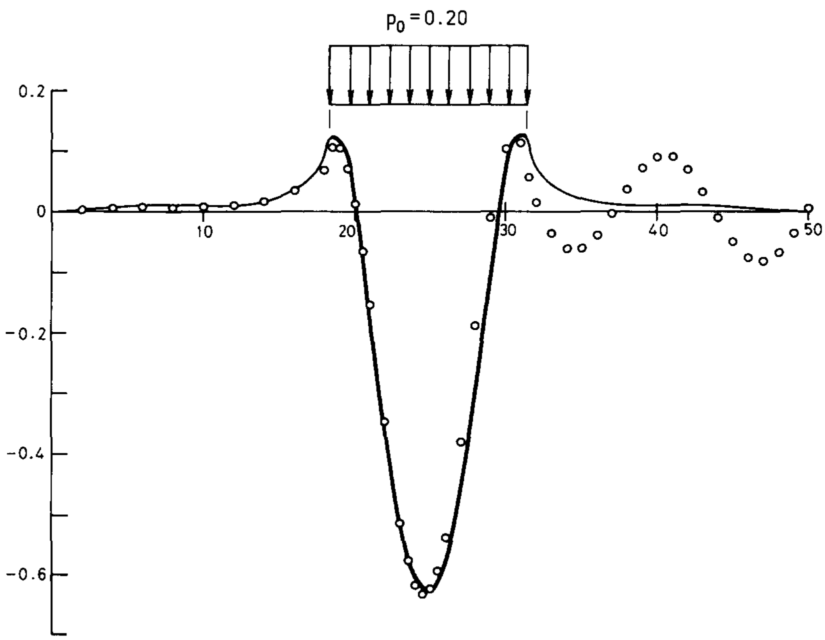

Figure 4. A nonlinear wave-free profile, $p_0 = .2$, $L = 13.00$, $Fr = .392$, —— numerical $N = 100$, ○ ○ ○ linear theory.

predicts simple sinusoidal behaviour from equation 14. The wave resistance is identically zero for $L = 4\pi$, 8π, ..., i.e. when the ship length is an integer multiple of a free wave length. For comparison two sets of nonlinear results are presented, for $p_0 = 0.1$ and 0.15. The general effects of nonlinearity are (i) to reduce the maximum resistance by between 5 and 10 per cent; (ii) to shift the curve to the right by an amount ΔL, which has a typical value of 0.5. These two effects are qualitatively similar to those reported in the finite depth calculation of von Kerczek and Salvesen [2]. The greatest absolute difference in C is about 0.2, and occurs for several different values of L.

An interesting question concerning wave resistance is whether a nonlinear solution with finite overpressure can ever have exactly zero drag. Figure 3 shows exceedingly small values of C_n near $L = 13.0$ and 27.8. By carefully adjusting the spacing h in order to vary L near the first critical point, it was possible to obtain the profile shown in Figure 4. This profile is symmetric about its minimum point to about 4 decimal places and the computed value of C_n is 7×10^{-5}. We expect that a more elaborate calculation can be made to produce still smaller values of C_n and can find no reason to doubt the general result that nonlinear wave-free profiles exist. The overpressure $p_0 = 0.2$ is a very large value in terms of the displacement it produces; yet this profile was obtained from an initially flat surface in about five iterations. We show the corresponding linear solution in the figure. It exhibits a small downstream wave train.

5. Concluding remarks

While the results of the present study are strictly valid only for two-dimensional flow problems, they may indicate qualitative trends in three-dimensional cases. To this extent, they would be applicable to hovercraft design. The present formulation can also be used to study the case of two (or more) isolated pressure distributions. It would thus be possible to estimate nonlinear interactive forces in the case of barges being towed in tandem. Clearly the total drag of such a configuration would be strongly affected by the separation distance and some optimization may be possible.

The extension to the finite depth case is straightforward. Equation (3), relating the real and imaginary parts of $z'(f)$ on $\psi = 0$ must be replaced by

$$x'(\phi) - \frac{2H}{\pi}\int_{-\infty}^{\infty} \frac{x'(\varphi)\,\mathrm{d}\varphi}{(\varphi-\phi)^2 + 4H^2} = -\frac{1}{\pi}\fint_{-\infty}^{\infty} y'(\varphi)\left[\frac{1}{\varphi-\phi} + \frac{(\varphi-\phi)}{(\varphi-\phi)^2 + 4H^2}\right]\mathrm{d}\varphi.$$

Here $\psi = -H$ on the horizontal bottom and the method of images has been used. Slight additional modifications will closely parallel the work of Vanden-Broeck & Schwartz [9] who compute periodic progressive waves in shallow water by a similar method. One additional matrix inversion will be required when H is finite. Solutions to the finite depth problem will allow direct comparison with the method of Von Kerczek and Salvesen [2] which, because it distributes grid points throughout the fluid region, cannot be applied to the infinite depth case.

With the enormous increase in computational capability that may be anticipated in the coming years, we have no doubt that the various methods of solution developed for two-dimensional free-surface problems will be viewed as candidates for extension to the more difficult and more important class of three-dimensional problems. It is our view that, because the present study has resulted in a very simple and straightforward technique, the problem of three-dimensional flow produced by a given overpressure and solved by a boundary-integral technique should be so considered.

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